The spinor representation of constant mean curvature one surfaces in the Hyperbolic space

Note di Matematica
Note Mat. 1(2008), suppl. n. 1, 317-339
ISSN 1123-2536, e-ISSN 1590-0932
DOI 10.1285/i15900932v28n1supplp317
http://siba2.unile.it/notemat

The spinor representation of CMC 1 surfaces in hyperbolic space

Emilio Musso[i]
Dipartimento di Matematica, Politecnico di Torino,
Corso Duca degli Abruzzi 24, I-10129 Torino, Italy
emilio.musso@polito.it

Lorenzo Nicolodi[ii]
Dipartimento di Matematica, Università degli Studi di Parma,
Viale G. P. Usberti 53/A, I-43100 Parma, Italy
lorenzo.nicolodi@unipr.it

Received: 12/12/07; accepted: 15/2/08.

Abstract. We review and comment on some aspects of the spinor representation for constant mean curvature one surfaces in hyperbolic space developed by Bobenko–Pavlyukevich–Springborn in [1]. The relations with the Bryant representation are addressed and some examples are discussed.

Keywords: Bryant surfaces, constant mean curvature one surfaces, isotropic curves, moving frames, spinor fields

MSC 2000 classification: primary 53A10

Introduction

This paper is concerned with the study of some aspects of the theory of conformal immersions of Riemann surfaces in hyperbolic 3-space whose mean curvature is constant equal to one (CMC 1 surfaces). In the seminal paper [4], Bryant showed that associated with any CMC 1 surface $f : S \to H^3$ there exists a multi-valued holomorphic null immersion Ψ of S into $\mathrm{SL}(2, \mathbb{C})$ and that the hyperbolic projection of such a Ψ is well defined on S and gives back the original f. This lead to a representation for CMC 1 surfaces in terms of holomorphic data which in general are not defined on the same Riemann surface as the conformal immersion.

After [4], the subject of CMC 1 surfaces has quickly taken hold as a rich and independent field of research (cf. [11]). Recently, Bobenko, Pavlyukevich and Springborn [1] have developed a representation for CMC 1 surfaces in terms

[i]This work is partially supported by the MIUR project “Metriche riemanniane e varietà differenziabili”

[ii]This work is partially supported by the MIUR project “Proprietà geometriche delle varietà reali e complesse”

of a pair of holomorphic spinor fields which are defined on the same Riemann surface as the immersion. (A similar approach was also used in the study of minimal surfaces in Euclidean space [9].) This representation allows the direct implementation of special functions in the study of CMC 1 surfaces and results in an extremely effective computational tool when discussing specific examples. Using this representation, Bobenko–Pavlyukevich–Springborn [1] clarified the geometry of known examples such as CMC 1 twonoids and derived explicit formulas for CMC 1 trinoids, which were implicitly classified by Umehara–Yamada using indirect methods [12].

Yet, several theoretical aspects of this spinor approach and the relations with the original construction of Bryant remain unclear (at least to these readers). Our purpose is to clarify these aspects by reformulating the original results of Bryant and Bobenko–Pavlyukevich–Springborn using the language of moving frames, the theory of flat connections on principal bundles and their holonomies. We explain why special functions can be effectively implemented within the theory of CMC 1 surfaces and discuss the structure underlying the examples considered in [1], [2] and [4]. In perspective, another purpose of this work is to provide a common background for the classical differential-geometric approach to CMC 1 surfaces and the algebraic-geometric one, recently developed by Pirola [10].

The paper is organized as follows. Section 1 collects some definitions and basic facts about spinors, suitably adapted to the purpose. Section 2 is devoted to the construction of the spinor structure and the pair of canonical spinor fields associated with any conformal immersion of a Riemann surface in H^3. Section 3 is concerned with the reformulation of the fundamental results of Bryant and Bobenko–Pavlyukevich–Springborn, and discusses the relations between CMC 1 surfaces and special functions. Section 4 discusses some examples, including twonoids and the related examples of Bohle and Peters [2], and the trinoids of Bobenko–Pavlyukevich–Springborn. Many explicit computations involving hypergeometric functions, as well as routine programs for the visualization of CMC 1 surfaces in hyperbolic space can be performed with MATHEMATICA.[1] For hypergeometric and other special functions we refer to [15], [7].

1 Preliminaries and notations

1.1 Some relevant geometry

The Minkowski 4-space $\mathbb{L}^4$ with the standard Lorentzian inner product

$$\langle\,,\rangle = -(x_0)^2 + (x_1)^2 + (x_3)^2 + (x_3)^2$$

[1]cf. the notebook at `http://www-sfb288.math.tu-berlin.de/~bobenko`.

is identified with the space $\mathcal{H}(2)$ of 2×2 Hermitian matrices by

$$(x_0, x_1, x_2, x_3) \in \mathbb{L}^4 \mapsto \begin{pmatrix} x_0 + x_3 & x_1 + ix_2 \\ x_1 - ix_2 & x_0 - x_3 \end{pmatrix} \in \mathcal{H}(2).$$

In this model, the Lorentzian inner product can be written as

$$\langle M, M \rangle = -\det M, \quad M \in \mathcal{H}(2).$$

As a consequence, hyperbolic 3-space H^3 can be identified with the space-like hypersurface

$$\{M \in \mathcal{H}(2) \,|\, \det M = 1,\, \mathrm{tr} M > 0\} \tag{1}$$

endowed with the induced metric. The group $\mathrm{SL}(2, \mathbb{C})$ acts isometrically and transitively on H^3 by

$$A \cdot M = AMA^*$$

for all $M \in \mathcal{H}(2)$, $A \in \mathrm{SL}(2, \mathbb{C})$, where A^* stands for the conjugate transpose of A. Then, the canonical projection

$$\pi : A \in \mathrm{SL}(2, \mathbb{C}) \mapsto AA^* \in H^3$$

gives $\mathrm{SL}(2, \mathbb{C})$ the structure of a principal bundle with structure group $\mathrm{SU}(2)$. Let $\Omega = A^{-1}dA$ be the holomorphic Maurer-Cartan form of $\mathrm{SL}(2, \mathbb{C})$ and $\Omega = \Omega_0 + \Omega_1$ its decomposition into skew-Hermitian and Hermitian parts given by

$$\begin{aligned} \Omega_0 &= \frac{1}{2}(\Omega - \Omega^*) = \frac{1}{2} \begin{pmatrix} i\omega_1^2 & \omega_1^3 + i\omega_2^3 \\ -\omega_1^3 + i\omega_2^3 & -i\omega_1^2 \end{pmatrix}, \\ \Omega_1 &= \frac{1}{2}(\Omega + \Omega^*) = \frac{1}{2} \begin{pmatrix} \omega_0^3 & \omega_0^1 + i\omega_0^2 \\ \omega_0^1 - i\omega_0^2 & -\omega_0^3 \end{pmatrix}. \end{aligned}$$

It follows that Ω_1 is semibasic for the projection π and that Ω_0 defines the *spinorial Levi-Civita connection* of H^3. In terms of the above decomposition, the structure equations of $\mathrm{SL}(2, \mathbb{C})$ amount to

$$d\Omega_0 + \Omega_0 \wedge \Omega_0 = -\Omega_1 \wedge \Omega_1, \quad d\Omega_1 = -\Omega_0 \wedge \Omega_1 - \Omega_1 \wedge \Omega_0. \tag{2}$$

1.2 Spin structures

A spin structure on a Riemann surface S is a holomorphic line bundle $\Sigma \to S$ such that $\Sigma \otimes \Sigma$ is the holomorphic cotangent bundle $\Lambda^{1,0}S$. A complex coordinate system $(\mathcal{U}, z)$ of S is said to be *admissible* if there exists a holomorphic section $\sqrt{dz} : \mathcal{U} \to \Sigma$ such that $\sqrt{dz} \otimes \sqrt{dz} = dz$. The triple $(\mathcal{U}, z, \sqrt{dz})$ will be called an *s-complex chart* on S. Spin structures do exist on any Riemann surface (cf. [9]).

The holomorphic sections of a spin structure are referred to as *holomorphic spinors*. If $\mathbf{P}$ and $\mathbf{Q}$ are holomorphic spinors, then their product $\mathbf{PQ}$ is a holomorphic 1-form. If $\mathbf{P}$ is a spinor and $(\mathcal{U}, z, \sqrt{dz})$ is an s-complex chart, we set $\mathbf{P}_{|\mathcal{U}} = P\sqrt{dz}$, where $P : \mathcal{U} \to \mathbb{C}$ is a holomorphic function, referred to as the *component* of $\mathbf{P}$ with respect to $(\mathcal{U}, z, \sqrt{dz})$.

2 The spin structure of an oriented surface in H^3

Let $f : S \to H^3$ be a conformal immersion of a connected Riemann surface S. The *bundle of zeroth order frame fields* along f is the principal SU(2)-bundle over S defined by

$$\mathcal{P}_0(f) = \{(p, A) \in S \times \mathrm{SL}(2, \mathbb{C}) : f(p) = AA^*\} \to S.$$

Let

$$\Sigma_f(S) = \mathcal{P}_0(f) \times_{\mathrm{SU}(2)} \mathbb{C}^2 \to S$$

be the associated $\mathbb{C}^2$-vector bundle. This is a topologically trivial rank-two complex vector bundle endowed with a Hermitian metric. The restriction of Ω_0 on $\mathcal{P}_0(f)$ is an SU(2)-connection which induces a Hermitian covariant derivative D_f on $\Sigma_f(S)$. As a consequence, there exists a unique holomorphic structure on $\Sigma_f(S)$ whose holomorphic sections are characterized by the fact that the $(0, 1)$-part of their covariant derivatives vanishes identically.[2]

Next, consider the unit normal vector field along f compatible with the orientation of S, i.e., the unique map

$$\mathcal{N} : S \to \mathcal{H}(2)$$

such that

$$\det \mathcal{N} = -1, \quad \langle \mathcal{N}, f\rangle = 0, \quad \langle \mathcal{N}, df\rangle = 0, \quad if \wedge \partial f \wedge \bar{\partial} f \wedge \mathcal{N} > 0. \qquad (3)$$

The *bundle of first order frame fields* along f is defined by

$$\pi_1 : \mathcal{P}_1(f) = \{(p, A) \in \mathcal{P}_0(f) : \mathcal{N}(p) = Ae_3A^*\} \to S,$$

where e_1, e_2, e_3 denotes the (complex conjugate) Pauli matrices

$$e_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad e_2 = \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix}, \quad e_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

[2]On a Riemann surface S, $\Lambda^2 S = \Lambda^{1,1} S$, so every Hermitian connection on a complex vector bundle over S defines a holomorphic structure. This is a standard fact in the theory of Hermitian vector bundles on a Riemann surface (cf. [8], [14]).

The structure group of $\mathcal{P}_1(f)$ is

$$e^{i\theta} \in \mathrm{U}(1) \to A(\theta) = \begin{pmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta} \end{pmatrix} \in \mathrm{SU}(2).$$

On $\mathcal{P}_1(f)$, the 1-form ω_0^3 vanishes identically, while the differential forms

$$\omega_0^1 \wedge \omega_0^2, \quad (\omega_0^1)^2 + (\omega_0^2)^2$$

are semibasic and invariant under the action of the structure group. Their projections to S give the area element $d\mathcal{A}_f$ and the first fundamental form $I_f = \langle df, df \rangle$ of the immersion, respectively. Moreover, the conformal property of f implies that the semibasic complex-valued 1-form

$$\omega := \omega_0^1 + i\omega_0^2$$

is of type $(1,0)$.[3] The off-diagonal term of Ω_0,

$$\upsilon = \frac{1}{2}(\omega_1^3 + i\omega_2^3)$$

is semibasic. Differentiating the equation $\omega_0^3 = 0$ yields

$$\upsilon \wedge \bar{\omega} - \omega \wedge \bar{\upsilon} = 0,$$

which implies the existence of smooth functions

$$h_{11}, h_{22}, h_{12} : \mathcal{P}_1(f) \to \mathbb{R}$$

such that

$$\upsilon = \frac{1}{4}(h_{11} + h_{22})\omega + \frac{1}{4}\left((h_{11} - h_{22}) + ih_{12}\right)\bar{\omega}.$$

Moreover, an easy inspection shows that

$$\langle d\mathcal{N}, df \rangle = h_{11}(\omega_0^1)^2 + 2h_{12}\omega_0^1\omega_0^2 + h_{22}(\omega_0^2)^2.$$

This means that the h_{ij}' are the coefficients of the second fundamental form of the immersion. The diagonal part of Ω_0

$$\rho = \frac{1}{2}\begin{pmatrix} i\omega_1^2 & 0 \\ 0 & -i\omega_1^2 \end{pmatrix}$$

defines a U(1)-connection on the bundle of first order frames, which is referred to as the *spinorial Levi-Civita connection* of I_f.

[3]This means that ω belongs to the ideal generated by $\pi_1^*(\Lambda^{1,0}S)$.

The first order frame bundle induces a splitting of Σ_f into the orthogonal direct sum of two rank one subbundles

$$\Sigma_f = \Sigma_f^+ \oplus \Sigma_f^-$$

defined by

$$\Sigma_f^+ = \mathcal{P}_1(f) \times_{r_+} \mathbb{C}, \quad \Sigma_f^- = \mathcal{P}_1(f) \times_{r_-} \mathbb{C},$$

where $r_\pm$ are the representations

$$r_+ : A(\theta) \in \mathrm{U}(1) \subset \mathrm{SU}(2) \to e^{i\theta}, \quad r_- : A(\theta) \in \mathrm{U}(1) \subset \mathrm{SU}(2) \to e^{-i\theta}.$$

The spinorial Levi-Civita connection defines Hermitian covariant derivatives D^+ and D^- on Σ_f^+ and Σ_f^-, respectively. On these line bundles, we then consider the corresponding holomorphic structures. Note that Σ_f^+ and Σ_f^- are not holomorphic sub-bundles of Σ_f.

Let $\widetilde{\Sigma}_f^+$ denote the dual bundle of Σ_f^+.

1 Proposition. *The holomorphic line bundle $\widetilde{\Sigma}_f^+ \to S$ defines a spin structure on the Riemann surface S.*

Proof. For a given first order frame field represented by the lift

$$\Phi = (\Phi_1, \Phi_2) : \mathcal{U} \to \mathrm{SL}(2, \mathbb{C}),$$

consider the nowhere vanishing $\Sigma_f^+ \otimes \Sigma_f^+$-valued 1-form of type $(1,0)$ locally defined by

$$\sigma|_\Phi = \Phi^*(\omega) \otimes (\Phi_1 \otimes \Phi_1).$$

The induced vector bundle isomorphism is denoted by

$$\widetilde{\sigma} : T^{1,0}S \to \Sigma_f^+ \otimes \Sigma_f^+.$$

If on $\Lambda^{1,0}S \otimes (\Sigma_f^+ \otimes \Sigma_f^+)$ we consider the tensor product of the Levi-Civita covariant derivative of $\Lambda^{1,0}S$ with the spinorial covariant derivative on Σ_f^+ it is easily seen that the section σ is parallel and hence $\widetilde{\sigma}$ is an isomorphism of holomorphic vector bundles. This yields the required result. $\boxed{QED}$

2 Definition. We call $\widetilde{\Sigma}_f^+$ the *canonical spin structure* of the conformal immersion f.

3 Remark. The holomorphic bundles $T^{1,0}S$ and $\Lambda^{1,0}S$ will be implicitly identified with $\Sigma_f^+ \otimes \Sigma_f^+$ and $\widetilde{\Sigma}_f^+ \otimes \widetilde{\Sigma}_f^+$, respectively, via the holomorphic isomorphisms constructed in the proof of the previous proposition.

Let $\Phi : \mathcal{U} \subset S \to \mathrm{SL}(2,\mathbb{C})$ be a first order frame field along f. The first column vector $\Phi_1 : S \to \mathbb{C}^2$ can be viewed as a section of the spinor bundle Σ_f^+. Under the above identification, $\Phi_1 \otimes \Phi_1$ is the nowhere vanishing $(1,0)$ differential form $\Phi^*(\omega)$. For brevity, we adopt the unconventional notation $\Phi_1 = \sqrt{\Phi^*(\omega)}$. Next, consider the two local sections $\mathbf{P}_{|\Phi}$ and $\mathbf{Q}_{|\Phi}$ of $\widetilde{\Sigma}_f^+$ defined by

$$\mathbf{P}_{|\Phi} = i\Phi_1^2\sqrt{\Phi^*(\omega)}, \quad \mathbf{Q}_{|\Phi} = i\Phi_1^1\sqrt{\Phi^*(\omega)}, \tag{4}$$

where Φ_1^1 and Φ_1^2 are the components of Φ_1. It is now easy to check that this definition is independent of the choice of first order frame fields. Thus there exist two global sections $\mathbf{P}_f$ and $\mathbf{Q}_f$ of $\widetilde{\Sigma}_f^+$ such that $\mathbf{P}_{f|\mathcal{U}} = \mathbf{P}_{|\Phi}$ and $\mathbf{Q}_{f|\mathcal{U}} = \mathbf{Q}_{|\Phi}$, for every first order frame field $\Phi : \mathcal{U} \to \mathrm{SL}(2,\mathbb{C})$.

4 Definition. We call $\mathbf{P}_f$ and $\mathbf{Q}_f$ the *Bobenko–Pavlyukevich–Springborn (BPS) spinor fields* of f.

5 Remark. From the definition it follows that $\mathbf{P}_f$ and $\mathbf{Q}_f$ have no common zeroes.

6 Remark. Although not invariant under the action of $\mathrm{SL}(2,\mathbb{C})$, the BPS spinor fields are equivariant in the sense that

$$\begin{pmatrix} \mathbf{P}(A\cdot f) \\ \mathbf{Q}(A\cdot f) \end{pmatrix} = (e_2 A e_2) \begin{pmatrix} \mathbf{P}(f) \\ \mathbf{Q}(f) \end{pmatrix}, \tag{5}$$

for every conformal immersion $f : S \to H^3$ and every $A \in \mathrm{SL}(2,\mathbb{C})$.

On the bundle of first order frames, consider the $\mathrm{SU}(2)$-valued 1-form given by

$$\theta = \Omega_0 + \frac{1}{2}\begin{pmatrix} 0 & -\omega \\ \bar{\omega} & 0 \end{pmatrix}. \tag{6}$$

The form θ uniquely extends to an $\mathrm{SU}(2)$-connection on the zeroth order frame bundle $\mathcal{P}_0(f)$. We will still denote this connection by θ.

7 Definition. We call θ the *Bryant connection* of the conformal immersion f. This modified connection is Hermitian and hence determines on Σ_f a new holomorphic structure.

3 The spinor approach to CMC 1 surfaces

3.1 The first basic theorem

Retaining the terminology and notation introduced above, we now examine the first basic result in the theory of CMC 1 surfaces (cf. [4], [1]).

8 Theorem. *Let S be a Riemann surface and $f : S \to H^3$ be a conformal immersion. Then the following statements are equivalent:*

(1) The mean curvature H_f is identically equal to one.

(2) The Bryant connection is flat.

(3) Σ_f^+ is a holomorphic line subbundle of Σ_f, where Σ_f is endowed with the holomorphic structure induced by the Bryant connection.

(4) The BPS spinor fields $\mathbf{P}_f$ and $\mathbf{Q}_f$ are holomorphic.

Proof. (1) $\Leftrightarrow$ (2). From the structure equation (2), it follows that the curvature of the Bryant connection is given by

$$d\theta + \theta \wedge \theta = \begin{pmatrix} i(1 - H_f) & 0 \\ 0 & -i(1 - H_f) \end{pmatrix} d\mathcal{A}_f. \tag{7}$$

This shows that the Bryant connection is flat if and only if $H_f \equiv 1$.

(1) $\Leftrightarrow$ (3). Notice that the line bundle Σ_f^+ is a holomorphic subbundle of Σ_f with the holomorphic structure induced by the Bryant connection if and only if the $(0,1)$-part of θ_1^2 vanishes identically. On the other hand, from the expression (6) of θ we have

$$\theta_1^2 = \frac{1}{2}(1 - H_f)\bar{\omega} - \frac{1}{4}\left[(h_{11} - h_{22}) - ih_{12}\right]\omega.$$

This implies that $H_f \equiv 1$ if and only if Σ_f^+ is a holomorphic subbundle of Σ_f (with the holomorphic structure induced by the Bryant connection).

(4) $\Leftrightarrow$ (3). By the definition (6) of the Bryant connection, it follows that

$$\Omega = \theta + J\omega,$$

where J is the nilpotent element of $\mathrm{SL}(2,\mathbb{C})$ given by $J = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$. We then have

$$d\Phi_1^1 = \theta_1^1\Phi_1^1 + \theta_1^2\Phi_2^1, \quad d\Phi_1^2 = \theta_1^1\Phi_1^2 + \theta_1^2\Phi_2^2, \quad D^+(\Phi_1) = -\theta_1^1 \otimes \Phi_1,$$

which implies

$$\begin{aligned} D^+(\mathbf{P}) &= iD^+(\Phi_1^2\Phi_1) = i\Phi_2^2\theta_1^2 \otimes \Phi_1, \\ D^+(\mathbf{Q}) &= iD^+(\Phi_1^1\Phi_1) = i\Phi_2^1\theta_1^2 \otimes \Phi_1. \end{aligned}$$

Since Φ_2^2 and Φ_2^1 have no common zeroes, it follows that $\mathbf{P}$, $\mathbf{Q}$ are holomorphic sections of $\widetilde{\Sigma}_f^+$ if and only if the $(0,1)$-part of θ_1^2 vanishes identically if and only if Σ_f^+ is a holomorphic subbundle of Σ_f. $\boxed{QED}$

3.2 The second basic theorem

9 Definition. Let $f : S \to H^3$ be a CMC 1 conformal immersion. By $\mathbf{P}_f$ and $\mathbf{Q}_f$, define the $\mathrm{SL}(2,\mathbb{C})$-valued 1-form

$$\nu(\mathbf{P}_f, \mathbf{Q}_f) = \begin{pmatrix} \mathbf{P}_f\mathbf{Q}_f & -\mathbf{Q}_f^2 \\ \mathbf{P}_f^2 & -\mathbf{P}_f\mathbf{Q}_f \end{pmatrix}. \tag{8}$$

This form is holomorphic and isotropic, in the sense that $\det \nu = 0$. Next, consider the associated linear system

$$d\Psi\Psi^{-1} = \nu(\mathbf{P}_f, \mathbf{Q}_f). \tag{9}$$

We call $\nu(\mathbf{P}_f, \mathbf{Q}_f)$ the *holomorphic isotropic differential* of f and refer to (9) as the *Bobenko–Pavlyukevich–Springborn (BPS) linear system* of f.

10 Definition. A zeroth order frame field $\Psi : \mathfrak{U} \subset U \to \mathrm{SL}(2,\mathbb{C})$ along f is called a *holomorphic isotropic lift* of f if Ψ is holomorphic and $\det(d\Psi) \equiv 0$.

The second main result of the theory is the following (cf. [4], [1]).

11 Theorem. *Let $f : S \to H^3$ be a CMC 1 conformal immersion. Then the following statements are equivalent:*

(1) $\Phi : \mathfrak{U} \to \mathrm{SL}(2,\mathbb{C})$ *is a parallel section of the Bryant connection.*

(2) $\Phi : \mathfrak{U} \to \mathrm{SL}(2,\mathbb{C})$ *is a holomorphic isotropic lift of f.*

(3) $\Phi : \mathfrak{U} \to \mathrm{SL}(2,\mathbb{C})$ *is a zeroth order frame field satisfying the BPS linear system of f.*

Proof. (1) $\Rightarrow$ (2). First, we prove that the parallel sections of the canonical connection are holomorphic isotropic lifts. Since this is a local property, we may assume that a parallel section $\Psi : \mathfrak{U} \subset S \to \mathrm{SL}(2,\mathbb{C})$ can be factorized as $\Psi = FA$, where $F : \mathfrak{U} \to \mathrm{SL}(2,\mathbb{C})$ is a first order frame field and $A : \mathfrak{U} \to \mathrm{SU}(2)$ is a smooth function. We then have

$$F^*(\theta) = F^{-1}dF - JF^*(\omega),$$

where J is the nilpotent element of $\mathrm{SL}(2,\mathbb{C})$ given by

$$J = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}.$$

From $\Psi^*(\theta) = 0$, we obtain

$$A^{-1}dA = -A^{-1}\left(F^{-1}dF - JF^*(\omega)\right)A,$$

which implies

$$\Psi^{-1}d\Psi = A^{-1}dA + A^{-1}F^{-1}dFA = A^{-1}JAF^*(\omega).$$

On the other hand, $F^*(\omega)$ is of type $(1,0)$ and $\det(J) = 0$, which implies that Ψ is holomorphic and isotropic.

(2) $\Rightarrow$ (1). Now, let $\Psi' : \mathcal{U} \subset S \to \mathrm{SL}(2,\mathbb{C})$ be a holomorphic isotropic lift. We may assume that $\Psi' = \Psi A$, where $\Psi : \mathcal{U} \to \mathrm{SL}(2,\mathbb{C})$ is a parallel lift and $A : \mathcal{U} \to \mathrm{SU}(2)$ is a smooth function. Since Ψ and Ψ' are holomorphic, then A is holomorphic too. But $\mathrm{SU}(2)$ is a real form of $\mathrm{SL}(2,\mathbb{C})$ and hence A is constant. This implies that Ψ' is parallel.

(2) $\Leftrightarrow$ (3). It is clear that if a zeroth order frame field Ψ satisfies the BPS linear system, then it is holomorphic and isotropic. Conversely, let $\Phi : \mathcal{U} \to \mathrm{SL}(2,\mathbb{C})$ be any zeroth order holomorphic section (a priori we do not need the assumption that Ψ is isotropic). Write $\Psi = \Phi A$, where $\Phi : \mathcal{U} \to \mathrm{SL}(2,\mathbb{C})$ is a first order frame and $A : \mathcal{U} \to \mathrm{SU}(2)$ is a smooth map. Differentiation of $\Psi = \Phi A$ yields

$$d\Psi\Psi^{-1} = \Phi\left(\Omega + dAA^{-1}\right)\Phi^{-1}. \tag{10}$$

Let

$$dAA^{-1} = \begin{pmatrix} i\alpha_1^1 & -\bar{\alpha}_1^2 \\ \alpha_1^2 & -i\alpha_1^1 \end{pmatrix},$$

where α_1^1 is real-valued. Since Ψ is holomorphic, the $(0,1)$-part of the left hand side of equation (10) vanishes identically. This implies that

$$i\alpha_1^1 = -\Omega_1^1, \quad \alpha_1^2 = \Omega_1^2.$$

We then have

$$d\Psi\Psi^{-1} = \Phi J\Phi^{-1}\omega.$$

Computing $\Phi J\Phi^{-1}$ we find

$$d\Psi\Psi^{-1} = \begin{pmatrix} -\Phi_1^1\Phi_1^2 & (\Phi_1^1)^2 \\ -(\Phi_1^2)^2 & \Phi_1^1\Phi_1^2 \end{pmatrix}\omega.$$

From (4), it follows that

$$\mathbf{P}_{|\Phi} = i\Phi_1^2\sqrt{\omega}, \quad \mathbf{Q}_{|\Phi} = i\Phi_1^1\sqrt{\omega},$$

and hence $d\Psi\Psi^{-1} = \nu(\mathbf{P}_f, \mathbf{Q}_f)$. $\boxed{QED}$

12 Remark. The equivalences of (1) and (2) in Theorem 8 and of (1) and (2) of Theorem 11 are essentially due to Bryant [4], while the equivalences of (1) and (4) in Theorem 8 and of (2) and (3) in Theorem 11 have been proved by Bobenko–Pavlyukevich–Springborn in [1]

3.3 Some comments on Bryant surfaces

An interesting class of CMC 1 surfaces in H^3 is that of the so-called *Bryant surfaces*, that is, CMC 1 conformal immersions defined on a punctured compact Riemann surface. One of the basic theorem of the theory (cf. [4]) affirms that every complete CMC 1 conformal immersion with finite total curvature is in fact a Bryant surface.

13 Proposition. *If $f : S \to H^3$ is a complete CMC 1 conformal immersion with finite total curvature, then $S = \tilde{S} \setminus \{p_1, \ldots, p_k\}$, where $\tilde{S}$ is a compact Riemann surface and $p_1, \ldots, p_k \in \tilde{S}$.*

As a consequence of the previous discussion, one sees that Bryant surfaces can be implicitly defined by two meromorphic spinor fields $(\mathbf{P}, \mathbf{Q})$ with no common zeroes on a compact Riemann surface S. The reconstruction of the immersion f goes as follows:

(1) Consider $|D| = \{p_1, \ldots, p_k\}$, the set of poles of the spinor fields.

(2) Solve the linear system $d\Psi\Psi^{-1} = \nu(\mathbf{P}, \mathbf{Q})$ on the universal covering $\hat{S}$ of $S \setminus |D|$.

(3) Then $\tilde{f} = \Psi\Psi^* : \hat{S} \to H^3$ is a Bryant surface whose spinor fields are the pull-backs of $\mathbf{P}$ and $\mathbf{Q}$ via the covering map of $\hat{S}$ onto $S \setminus |D|$.

However, in general $\tilde{f}$ is not invariant under the deck transformations of the universal covering and henceforth it does not defines a Bryant immersion. From a conceptual viewpoint, let start from a pair of meromorphic spinor fields $(\mathbf{P}, \mathbf{Q})$ with no common zeroes on a compact Riemann surface $\hat{S}$, and let $\Delta(\mathbf{P}, \mathbf{Q})$ denote the set of the poles of $\mathbf{P}$ and $\mathbf{Q}$. Then the solutions of the linear system

$$d\Psi\Psi^{-1} = \nu(\mathbf{P}, \mathbf{Q})$$

originate an $\mathrm{SL}(2, \mathbb{C})$-valued sheaf on $S = \hat{S} \backslash \Delta(\mathbf{P}, \mathbf{Q})$ whose transition functions are locally constant. This means that this sheaf generates an $\mathrm{SL}(2, \mathbb{C})$-principal bundle

$$B(\mathbf{P}, \mathbf{Q}) \to S \tag{11}$$

endowed with a flat $\mathrm{SL}(2, \mathbb{C})$-connection $\beta(\mathbf{P}, \mathbf{Q})$.

14 Definition. The pair $(\mathbf{P}, \mathbf{Q})$ is said *unitarizable* if the holonomy group $\mathrm{Hol}(\mathbf{P}, \mathbf{Q}, p)$ of $\beta(\mathbf{P}, \mathbf{Q})$ (computed with respect to some base point $p \in S$) is conjugate to a subgroup of $\mathrm{SU}(2)$.

15 Proposition. *The pair $(\mathbf{P}, \mathbf{Q})$ arises from a Bryant immersion if and only if it is unitarizable.*

Proof. Suppose that there exists $R \in \mathrm{SL}(2, \mathbb{C})$ such that

$$R\mathrm{Hol}(\mathbf{P}, \mathbf{Q}, p)R^{-1} \subset \mathrm{SU}(2).$$

Consider the holonomy bundle

$$B_h(\mathbf{P}, \mathbf{Q}, p) \to S$$

of $\beta(\mathbf{P}, \mathbf{Q})$ with respect to the base point $p \in S$. Two parallel local cross sections are represented by holomorphic maps

$$\Psi : \mathcal{U} \subset S \to \mathrm{SL}(2, \mathbb{C}), \quad \Psi' : \mathcal{U}' \subset S \to \mathrm{SL}(2, \mathbb{C})$$

and the corresponding transition function is a locally constant map

$$A : \mathcal{U} \cap \mathcal{U}' \to \mathrm{Hol}(\mathbf{P}, \mathbf{Q}, \mathrm{p}).$$

This implies that if we denote by $\{(\mathcal{U}_j, \Psi_j)\}_{j \in J}$ the collection of local parallel sections of $B_h(\mathbf{P}, \mathbf{Q}, p)$, then the transition functions of the modified family $\{(\mathcal{U}_j, \Psi_j R^{-1})\}$ take values in $\mathrm{SU}(2)$. Thus there exists a well-defined Bryant immersion $f : S \to H^3$ such that

$$f_{|\mathcal{U}_j} = \Psi_j R^{-1}(R^{-1})^* \Psi_j^*, \quad \forall j \in J.$$

It is clear that $\mathbf{P}$ and $\mathbf{Q}$ coincide with the two BPS spinor fields of f. The converse can be proved by similar arguments. $\boxed{QED}$

16 Remark. Note that $(\mathbf{P}, \mathbf{Q})$ is a unitarizable pair if and only if the induced covariant derivative on the $\mathbb{C}^2$-vector bundle $B(\mathbf{P}, \mathbf{Q}) \times_{\mathrm{SL}(2,\mathbb{C})} \mathbb{C}^2$ admits a parallel Hermitian metric.

From the above discussion it follows that the study of Bryant surfaces in H^3 can ultimately be reduced to the study of unitarizable pairs of meromorphic spinor fields on compact Riemann surfaces.

3.4 Bryant surfaces and special functions

In this section we use the spinor approach in a more concrete fashion. The question is how to produce Bryant immersions starting from a pair of meromorphic spinor fields $(\mathbf{P}, \mathbf{Q})$ on the Riemann sphere $S^2 = \mathbb{C} \cup \{\infty\}$. The following remark turns out to be important for the construction of explicit examples.

17 Remark. Consider the linear system of ODE

$$\Phi' = L\Phi \tag{12}$$

where $L : S^2 \to \mathrm{SL}(2,\mathbb{C})$ is a meromorphic map given by

$$L = \begin{pmatrix} a & b \\ c & -a \end{pmatrix},$$

being a, b, c meromorphic functions on S^2. The components of either column vector $(x, y)^T$ of a solution Φ are linearly independent solutions of the system

$$x' = ax + by, \quad y' = cx - ay.$$

If $c \neq 0$, then

$$x = \frac{1}{c}(y' + ay)$$

and substitution into the first equation yields the second order linear ODE

$$\mathcal{S}_+(y, L) = y'' - \frac{c'}{c}y' + \left(\frac{a'c - ac'}{c} - (a^2 + cb)\right) y = 0. \tag{13}$$

Interchanging the role of the two variables, if $b \neq 0$, then x must satisfy

$$\mathcal{S}_-(x, L) = x'' - \frac{b'}{b}x' + \left(\frac{ab' - a'b}{b} - (a^2 + cb)\right) x = 0. \tag{14}$$

Therefore, the general solutions $\Phi : \mathcal{U} \subset S^2 \to \mathrm{SL}(2,\mathbb{C})$ of (12) can be obtained from a pair of linearly independent solutions of (13) or of (14). In fact, contemplate fundamental local solutions $(h_1^\pm, h_2^\pm) : \mathcal{U} \to S^2$ of (13) and (14), respectively, and put

$$\begin{aligned} M_+ &= \begin{pmatrix} \frac{1}{c}\left(\frac{dh_1^+}{dz} + ah_1^+\right) & \frac{1}{c}\left(\frac{dh_2^+}{dz} + ah_2^+\right) \\ h_1^+ & h_2^+ \end{pmatrix}, \\ M_- &= \begin{pmatrix} h_1^- & h_2^- \\ \frac{1}{b}\left(\frac{dh_1^-}{dz} - ah_1^-\right) & \frac{1}{b}\left(\frac{dh_2^-}{dz} - ah_2^-\right) \end{pmatrix}. \end{aligned}$$

It is easily checked that $\det M_\pm$ is a non-zero constant and, more importantly, that the general solutions of (12) are of the form

$$\Phi_\pm(h_1^\pm, h_2^\pm, A) = \frac{1}{\sqrt{\det M_\pm}} M_\pm A, \tag{15}$$

for some $A \in \mathrm{SL}(2,\mathbb{C})$.

We now apply these observations to our specific situation. Given a pair $(\mathbf{P}, \mathbf{Q})$ of meromorphic spinors on S^2, write $\mathbf{P} = P\sqrt{dz}$ and $\mathbf{Q} = Q\sqrt{dz}$, where P and Q are meromorphic functions. Then the local problem amounts to integrating the linear system

$$\Psi' = \begin{pmatrix} PQ & -Q^2 \\ P^2 & -PQ \end{pmatrix} \Psi.$$

The previous remark tells us how to recover its solutions from the second order ODE

$$\frac{d^2h}{dz^2} - \frac{2}{P}\frac{dP}{dz}\frac{dh}{dz} - \left(\frac{dP}{dz}Q - P\frac{dQ}{dz}\right) h = 0 \tag{16}$$

when $P^2 \neq 0$, or the analogous equation when $Q^2 \neq 0$. For special choices of $\mathbf{P}$ and $\mathbf{Q}$ one can integrate (16) in terms of special functions. Assume we are in this case (otherwise, there is no hope to explicitly solve the problem). Let $\Delta = \{p_1, \dots, p_k\}$ be the set of poles of the spinor fields, then the local solutions h_1 and h_2 around p_j are special functions defined on $\mathbb{C} \cup \{\infty\}$ minus a cut C_j, that is a curve starting from p_j and not containing the other poles of the spinor fields. Thus, one can explicitly produce a *complete family* $\mathcal{F} = \{\mathcal{U}_j, \Psi_j\}_{j \in \{1,\dots,k\}}$ *of local solutions*.[4] If a complete family $\mathcal{F}$ has been constructed, then the *unitarizability problem* can be solved as follows:

- Take a small circle around the pole p_j, say

  $$\gamma_j(t) = r_j(\cos(\theta) + i\sin(t)) - p_j,$$

  and choose the radius so that the circle intersects the cut C_j only in one point, say $\gamma_j(t_j)$.

- Then compute the two limits

  $$B_j^+ = \lim_{s \to 0^+} \Psi_j[\gamma_j(t_j + s)], \quad B_j^- = \lim_{s \to 0^-} \Psi_j[\gamma_j(t_j + 2\pi + s)].$$

- Define the corresponding *monodromy matrix* $A_j \in \mathrm{SL}(2, \mathbb{C})$ by

  $$A_j = (B_j^+)^{-1} B_j^-.$$

  The set $\mathcal{A} = \{A_1, \dots, A_k\}$ gives the *monodromy data* of the family $\mathcal{F}$.

18 Definition. We say that a complete family $\mathcal{F}$ of local solutions is *unitarizable* if there exists $R \in \mathrm{SL}(2, \mathbb{C})$ such that $R^{-1} A_j R \in \mathrm{SU}(2)$, $j = 1, \dots, k$. The matrix R is said to be a *unitarization* of the family.

[4] The term "complete" simply means that $S = \bigcup_j \mathcal{U}_j$.

If $\mathcal{F}$ is unitarizable and if R is a unitarization, then the *modified family*

$$\mathcal{F}R = \{(\mathcal{U}_j, \Psi_j R)\}$$

originates a well-defined Bryant immersion $f : S^2 \setminus \Delta \to H^3$ such that

$$f_{|\mathcal{U}_j} = \Psi_j R R^* \Psi_j^*, \quad j = 1, \dots, k.$$

Moreover $\mathbf{P}$ and $\mathbf{Q}$ are exactly the spinor fields of f. By this procedure one can construct Bryant surfaces with prescribed spinor fields.

The final problem is to understand the moduli space of the Bryant surfaces with assigned spinor fields. The appropriate equivalence relation is the obvious one.

19 Definition. Two Bryant immersions $f, f' : S \to H^3$ are said *equivalent* if there exist a biholomorphic map $h : S \to S$ and an element $A \in \mathrm{SL}(2, \mathbb{C})$ such that $f' = A(f \circ h)A^*$.

20 Remark. The problem of producing an explicit family of local solutions of (16) can be rather tricky. In fact, it may happens that (16) is not directly solvable. In this case, one has to find an explicit holomorphic map $G : S \setminus \Delta \to \mathrm{PSL}(2, \mathbb{C})$[5] such that the second order differential equation defined by the modified linear system

$$\Phi' = G\left(\nu(P, Q) + G^{-1}G'\right)G^{-1}\Phi, \tag{17}$$

can be solved in term of special functions. The most difficult point in the whole construction is the determination of a unitarizable family. In general, unitarizable families seems to be rather special and in most cases do not exist at all (also when (16) or a gauge-equivalent equation can be solved explicitly). The analysis of this problem is intrinsically related to the study of the monodromies of Fuchsian equations which by itself represents an interesting and rich field of research (cf. [13] and the bibliography therein).

21 Remark. Of particular interest in the applications are the spinor pairs $(\mathbf{P}, \mathbf{Q})$ of *hypergeometric type*, i.e., meromorphic spinors on S^2 for which it is possible to construct an explicit gauge transformation

$$G : S^2 \setminus \Delta \to \mathrm{PSL}(2, \mathbb{C})$$

such that the second order ODE defined by (17) takes the form

$$\frac{d^2y}{dz^2} + \left(\frac{1-a-a'}{z} + \frac{1-b-b'}{z-1}\right)\frac{dy}{dz} + \left(\frac{bb'}{z-1} - \frac{aa'}{z} + cc'\right)\frac{y}{z(z-1)} = 0, \tag{18}$$

[5]Here $\mathrm{PSL}(2, \mathbb{C})$ is the projective group $\mathrm{SL}(2, \mathbb{C})/\mathbb{Z}_2$, where $\mathbb{Z}_2$ is the center of $\mathrm{SL}(2, \mathbb{C})$.

where the coefficients satisfy the relation

$$a + a' + b + b' + c + c' = 1.$$

In this case, (18) can be explicitly solved in terms of confluent hypergeometric functions. To our knowledge, not much is known about unitarizable hypergeometric spinor pairs. Explicit examples have been carefully analyzed by Bobenko–Pavlyukevich–Springborn in [1], but it is not clear if they exhaust the family of all unitarizable spinor pairs of hypergeometric type. We shall discuss these example in the next section.

22 Remark. A rather exceptional class is that of *exact* spinor pairs, that is meromorphic spinors $(\mathbf{P}, \mathbf{Q})$ on S^2 for which there exists an explicit gauge transformation $G : S^2 \setminus \Delta \to \mathrm{PSL}(2, \mathbb{C})$ such that the second order ODE defined by (17) admits two linearly independent meromorphic solutions defined on all S^2. For an exact pair, the linear system admits a univalent meromorphic solutions $\Psi : S^2 \to \mathrm{SL}(2, \mathbb{C})$. The only known examples are related to the "integral" twonoids and will be analyzed in the next section.

23 Remark. In the next section, we show how this approach has been successfully applied to Bryant immersions of the punctured sphere, when the spinor fields have two or three simple poles.

4 Examples

4.1 Twonoids (cf. [4], [1])

A twonoid is a Bryant surface defined on $S^2 \setminus \{0, \infty\}$ by a pair of spinor fields of the form

$$\mathbf{P} = \left(\frac{p_0}{z} + p_\infty\right)\sqrt{dz}, \quad \mathbf{Q} = \left(\frac{q_0}{z} + q_\infty\right)\sqrt{dz}.$$

Using the $\mathrm{SL}(2, \mathbb{C})$-equivariance and a reparametrization of the form $z \to az$, one can assume that

$$p_\infty = q_0 = 0, \quad p_0 = q_\infty = i\sqrt{m(m+1)},$$

where

$$m \in \mathbb{C} \setminus \{0, -1\}.$$

By (14), the corresponding second order differential equation becomes

$$y'' - m(m+1)z^{-2}y = 0.$$

Under the additional assumption $m \neq -1/2$, two linearly independent solutions of the equation are given by the multi-valued functions $h_1(z) = z^{m+1}$ and $h_2(z) = z^{-m}$. According to Remark 17, it follows that the isotropic lifts can be represented, up to right multiplication by an element of $\mathrm{SL}(2,\mathbb{C})$, by the multi-valued map

$$\Psi_m = \frac{1}{\sqrt{-2m-1}} \begin{pmatrix} mz^{m+1} & (m+1)z^{-m} \\ (m+1)z^m & mz^{-(m+1)} \end{pmatrix}. \tag{19}$$

The monodromy of Ψ_m around the origin is

$$\mathcal{M}_m = \begin{pmatrix} e^{-2m\pi i} & 0 \\ 0 & e^{2m\pi i} \end{pmatrix},$$

which is unitary (and unitarizable) if and only if $m \in \mathbb{R}$. This explains the reason why, for real values of the parameter m, the hyperbolic projections of the multi-valued maps Ψ_m generate well defined Bryant immersions

$$f_m : S^2 \setminus \{0, \infty\} \to H^3,$$

also known as the *catenoid cousins*. A short computation shows that

$$f_m(z) = -\frac{1}{2m+1} X_m(z) \begin{pmatrix} |z|^{2m} & 0 \\ 0 & |z|^{-2m} \end{pmatrix} X_m(z)^*,$$

where

$$X_m(z) = \begin{pmatrix} mz & m+1 \\ m+1 & mz^{-1} \end{pmatrix}.$$

These are surfaces of revolution, whose profile curve is embedded for $m \in (-\frac{1}{2}, 0)$, and has a simple self-intersection otherwise.

Now, a natural question arises whether there exist twonoids which are not equivalent to a catenoid cousin. This can happen only if m is half-integer. In this case $\mathcal{M}_m = \pm I$ and therefore, taking any non-diagonal $A \in H^3 \subset \mathcal{H}(2)$ the hyperbolic projection of the multi-valued isotropic curve $\Psi_m A$ originates a well-defined Bryant immersion denoted by

$$A \sharp f_m : S^2 \setminus \{0, \infty\} \to H^3.$$

These new examples are not equivalent (neither isometric) to a catenoid cousin.

24 Remark. It is worth observing that the catenoid cousins have a continuous symmetry group (i.e., the rotational group around the axis of revolution). The other twonoids do not enjoy this property. However, they admit a non-trivial finite group of symmetries. More precisely,

- if the parameter m is an integer, then they have a finite symmetry group isomorphic to $\mathbb{Z}_2 \times \mathbb{Z}_{2m+1}$;
- if $m = h + 1/2$ (with $h \in \mathbb{Z}$) is half-integer, then the corresponding symmetry group is isomorphic to $\mathbb{Z}_2 \times \mathbb{Z}_{2h}$.

Another remarkable geometric property satisfied by all twonoids is that they have *smooth ends*. The precise meaning of the term "smooth end" is explained in the following.

25 Definition. Let $f : S \setminus D \to H^3$ be a Bryant immersion defined on a punctured Riemann surface S and consider the Poincaré model for the hyperbolic space (i.e., H^3 is viewed as the unit ball of $\mathbb{R}^3$). Then $p \in D$ is said to be a smooth end if $f : S \setminus D \to H^3 \subset \mathbb{R}^3$ extends smoothly in a neighborhood $\mathcal{U} \subset S$ of p.

As observed in [4], the catenoid cousins f_m have finite total curvature $\mathcal{K}(f_m)$, namely

$$\mathcal{K}(f_m) = \int K(f_m) d\mathcal{A}_m = -4\pi(2m+1),$$

where $K(f_m)$ is the Gaussian curvature of f_m and $d\mathcal{A}_m$ is the area element. Observe that

$$-\frac{\mathcal{K}(f_m)}{4\pi} \in \mathbb{Z} \quad \Longleftrightarrow \quad m \in \frac{1}{2}\mathbb{Z} \quad \Longleftrightarrow \quad f_m \quad \text{is half-integral},$$

where we adopt the following terminology.

26 Definition. A CMC 1 immersion $f : S \to H^3$ is said *integral*, respectively *half-integral*, if it admits a well-defined holomorphic isotropic lift with values in $\mathrm{SL}(2,\mathbb{C})$, respectively $\mathrm{SL}(2,\mathbb{C})/\mathbb{Z}_2$.

27 Remark. Integral CMC 1 surfaces are characterized by the fact that the flat connection on $B(\mathbf{P}_f, \mathbf{Q}_f)$ is trivial, while half-integral CMC 1 surfaces are characterized by the fact that the holonomy group is contained in $\mathbb{Z}_2$.

4.2 The Bohle-Peters examples and the "integral" twonoids

Taking into account that the property for a holomorphic curve in $\mathrm{SL}(2,\mathbb{C})$ of being isotropic is invariant under the full pseudo-group of holomorphic conformal transformations, and observing that integral twonoids do admit globally defined holomorphic isotropic lifts into $\mathrm{SL}(2,\mathbb{C})$, Bohle and Peters [2] were able to construct immersed (non embedded) Bryant surfaces with an arbitrary even number of smooth ends. Their construction goes as follows: let m be an integer and let s be a non-zero complex number. Then, let $F_s : \mathrm{SL}(2,\mathbb{C}) \to \mathrm{SL}(2,\mathbb{C})$ be the conformal rational map defined by

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \to \rho \begin{pmatrix} a & 2s(1-s^2) - s^2c + (1-s^2)^2 b \\ c + 2s - s(1-s^2)b & d \end{pmatrix},$$

where

$$\rho = \frac{1}{(1-2s^2) - sc - s(1-s^2)b}$$

and let

$$\Psi_m : S^2 \setminus \{0,\infty\} \to \mathrm{SL}(2,\mathbb{C})$$

be the holomorphic isotropic curve defined as in (19). Then, the hyperbolic projections of the isotropic curves

$$\Psi_{s,m,A} = (F_s \circ \Psi_m)A : S^2 \setminus \{0,\infty\} \to \mathrm{SL}(2,\mathbb{C}),$$

where $m \in \mathbb{Z}$ and $A \in H^3$, provide examples of (non-embedded) Bryant surfaces with an an arbitrary even number of smooth ends; some of these examples have cyclic symmetry groups.

The construction of the Bohle–Peters examples is based on the fact that the holomorphic conformal compactification of $\mathrm{SL}(2,\mathbb{C})$ is the smooth complex quadric $Q^3 \subset \mathbb{CP}^4$ defined by

$$z_0^2 + z_1^2 + z_2^2 + z_3^2 + z_4^2 = 0.$$

The conformal embedding of $\mathrm{SL}(2,\mathbb{C})$ into Q^3 is given by the affine chart

$$(a^i_j) \in \mathrm{SL}(2,\mathbb{C}) \mapsto \left[\left(-\frac{i(a^1_1 + a^2_1)}{2}, \frac{a^1_1 - a^2_2}{2}, -\frac{i(a^1_2 - a^2_1)}{2}, \frac{a^2_1 + a^1_2}{2}, 1\right)\right]$$

with inverse

$$\mathrm{sp} : [(x_0, x_1, x_2, x_3, x_4)] \in Q^3 \mapsto \begin{pmatrix} \dfrac{i(x_0 + ix_1)}{x_4} & \dfrac{x_2 + ix_3}{x_4} \\ \dfrac{x_2 - ix_3}{x_4} & \dfrac{i(x_0 - ix_1)}{x_4} \end{pmatrix}.$$

Note that the "ideal boundary" of $\mathrm{SL}(2,\mathbb{C})$ in Q^3 is the quadric Q^2_∞ defined by

$$z_0^2 + z_1^2 + z_2^2 + z_3^2 = 0, \quad z_4 = 0.$$

Therefore, if

$$\hat{\Psi} : S \to Q^3$$

is a null curve not contained in the ideal boundary, then

$$\Psi' = \mathrm{sp} \circ \hat{\Psi} : S \setminus D \to \mathrm{SL}(2, \mathbb{C})$$

is an isotropic curve of $\mathrm{SL}(2, \mathbb{C})$ defined on the complement of the divisor

$$D = \{p \in S \ : \ \hat{\Psi}(p) \in Q^2_\infty\}.$$

Thus, if

$$\Psi : S \setminus D \to \mathrm{SL}(2, \mathbb{C})$$

is an isotropic curve defined on a punctured compact Riemann surface $S = \hat{S} \setminus D$ which completes to a null curve

$$\hat{\Psi} : \hat{S} \to Q^3$$

and if $T \in \mathrm{O}(5, \mathbb{C})$ is a conformal transformation of Q^3, then

$$\mathrm{sp}(T \circ \hat{\Psi}) : \hat{S} \setminus D_T \to \mathrm{SL}(2, \mathbb{C})$$

is an isotropic curve defined on the complement of the divisor

$$D_T = \{p \in S : T \circ \hat{\Psi}(p) \in Q^2_\infty\}.$$

If $f : S \setminus D \to H^3$ is a Bryant immersion that admits a well-defined isotropic lift $\Psi : S \setminus D \to \mathrm{SL}(2, \mathbb{C})$ and if this lift extends to a null curve $\hat{\Psi} : \hat{S} \to Q^3$, then, for each conformal transformation $T \in \mathrm{O}(5, \mathbb{C})$ and for every $A \in H^3$, the mapping

$$\mathrm{sp}(T \circ \hat{\Psi}) A A^* \mathrm{sp}(T \circ \hat{\Psi})^* : \hat{S} \setminus D_T \to H^3$$

defines a new Bryant immersion which in general is not equivalent to the original one. This provides a theoretical explanation of the above examples. We can also understand why the Bohle–Peters examples have an even number of smooth ends: for an integral value of the parameter m the projective completion $\mathcal{G}_m \subset Q^3$ of the isotropic curve Ψ_m intersects the ideal boundary Q^2_∞ at the points $[(0, 0, -im, m, 0)]$ and $[(0, 0, im, m, 0)]$ with a contact of order $(m + 1)$. Thus, when we transform $\mathcal{G}_m \subset Q^3$ via a generic conformal map $T \in \mathrm{O}(5, \mathbb{C})$ (such as the one considered by Bohle–Peters), the new curve $T\mathcal{G}_m$ intersects Q^2_∞ in $2(m + 1)$ distinct points. This implies that the Bryant immersion defined by $T\mathcal{G}_m$ has exactly $2(m + 1)$ smooth ends.

28 Remark. In this context one can show that the isotropic curves of the integral twonoids are classical W-curves in Q^3. Moreover, as indicated by [2], this observations establish a link between Bryant surfaces with smooth ends and Willmore isothermic surfaces and superminimal surfaces in S^4 [3, 6]. The general structure of the moduli space of isotropic curves in Q^3 is a classical unsolved problem in algebraic geometry. Also in the case of rational isotropic curves one has few results in low degrees [5].

4.3 Trinoids

Using the spinor representation, Bobenko, Pavlyukevich and Springborn [1] derived explicit formulas for CMC 1 trinoids. In this section we overview the conceptual scheme behind the construction without any pretension of completeness as regards the subtle technical details: for these we refer to [1].

In these examples the holomorphic data are:

- The Riemann sphere punctured at three points
$$S = S^2 \setminus \{0, 1, \infty\}.$$
- Two meromorphic spinors of the form
$$\mathbf{P} = \left(\frac{p_0}{z} + \frac{p_1}{z-1} + p_3\right)\sqrt{dz}, \quad \mathbf{Q} = \left(\frac{q_0}{z} + \frac{q_1}{z-1} + q_3\right)\sqrt{dz}. \tag{20}$$

The second order ODE (16) is not directly in hypergeometric form. To put the system in the required form, one has to perform a gauge transformation defined by means of a holomorphic map

$$G : z \in S \to \left[G_1(z) \cdot G_2(z) \cdot \begin{pmatrix} 2a/\lambda & 0 \\ 1 & 1 \end{pmatrix}\right] \in \mathrm{SL}(2, \mathbb{C})/\mathbb{Z}_2,$$

where G_1 and G_2 are defined by

$$G_1(z) = \begin{pmatrix} iQ(z) & a_1 z + b_1 \\ iP(z) & a_2 z + b_2 \end{pmatrix}, \quad G_2(z) = \begin{pmatrix} \sqrt{z-1} & 0 \\ \frac{k}{z\sqrt{z-1}} & \frac{1}{\sqrt{z-1}} \end{pmatrix}.$$

One can determine the coefficients a_1, a_2, b_1, b_2, k, a and λ in terms of the residues p_0, p_1, p_∞ and q_0, q_1, q_∞, so that the local solutions Ψ of (9) are of the form $\Phi = G \cdot \Psi$, where Φ satisfies the modified linear system

$$\Phi' = \left(\frac{1}{z}\begin{pmatrix} \alpha & 0 \\ 0 & -\alpha \end{pmatrix} + \frac{1}{z-1}\begin{pmatrix} \beta & \gamma \\ \delta & -\beta \end{pmatrix}\right)\Phi, \tag{21}$$

and α, β, γ and δ explicitly depend on p_0, p_1, p_∞ and q_0, q_1, q_∞.[6] It is now easy to check that (21) is equivalent to the hypergeometric equation

$$\frac{d^2y}{dz^2} + \left(\frac{\alpha}{z} + \frac{1+2\beta}{z-1}\right)\frac{dy}{dz} + \left(\frac{\beta - \delta\gamma}{z-1} + \frac{\alpha}{z} + \beta + \gamma\right)\frac{y}{z(z-1)} = 0,$$

[6] It is still mysterious to us how Bobenko, Pavlyukevich and Springborn were able to guess the appropriate form of the gauge transformation.

which can be explicitly solved in terms of confluent hypergeometric functions (one can write the local solutions using MATHEMATICA). As a consequence, one can explicitly construct a complete family of local solutions to $\Psi' = \nu(P,Q)\Psi$. Since there are three singularities, this family contains three elements. To check the unitarizability is the most subtle task. The monodromy matrices of the family can be computed explicitly in terms of the values $\Gamma(w_0), \Gamma(w_1)$ and $\Gamma(w_2)$ of the Gamma function at three points w_0, w_1 and w_2, which can be determined in terms of the residues of the spinors. As a result, Bobenko, Pavlyukevich and Springborn showed that the unitarizable spinors depend on three real parameters, and hence the moduli space of Bryant conformal immersions with holomorphic data (20) is 3-dimensional. It is remarkable that this space contains a 1-parameter family $\{f_\mu\}_{\mu\in\mathbb{R}}$ of trinoids possessing a trihedral symmetry group. They also showed that there exists $\mu_0 > 1/4$ such that, for $\mu < \mu_0$, all symmetric trinoids f_μ are embedded, while, for $\mu > \mu_0$, they are not embedded. All the computations have been implemented in a MATHEMATICA notebook that can be found at `http://www-sfb288.math.tu-berlin.de/~bobenko`.

References

[1] A. I. Bobenko, T. V. Pavlyukevich, B. A. Springborn: Hyperbolic constant mean curvature one surfaces: Spinor representation and trinoids in hypergeometric functions, *Math. Z.* **245** (2003), 63–91; preprint available as `arXiv:math.DG/0206021`.

[2] C. Bohle, G. Paul Peters: *Bryant surfaces with smooth ends*, preprint available as `arXiv:math.DG/0411480v2`.

[3] R. L. Bryant: A duality theorem for Willmore surfaces, *J. Differential Geom.* **20** (1984), 23–53.

[4] R. L. Bryant: Surfaces of mean curvature one in hyperbolic space, Théorie des variétés minimales et applications (Palaiseau, 1983–1984), *Astérisque* **154-155** (1987), 321–347.

[5] R. L. Bryant: Surfaces in conformal geometry, *Proc. Sympos. Pure Math.* **48** (1988), 227–240, Amer. Math. Soc., Providence, RI.

[6] S.-S. Chern: On minimal spheres in the four-sphere, in: Studies and Essays (presented to Yu-Why Chen on his 60th Birthday, April 1, 1970) Math. Res. Center, Nat. Taiwan Univ, Taipei, 1970, pp. 137–150.

[7] A. Erdélyi, W. Magnus, F. Oberhettinger, F. Tricomi: Higher transcendental functions. Vols. I, II, McGraw-Hill Book Company, Inc., New York, Toronto, London, 1953.

[8] P. Griffiths, J. Harris: *Principles of algebraic geometry*, Pure and Applied Mathematics, Wiley-Interscience, New York, 1978.

[9] R. Kusner, N. Schmitt: The spinor representation of surfaces in space, preprint available as `arXiv:dg-ga/9610005`.

[10] G. P. Pirola: Monodromy of constant mean curvature surface in hyperbolic space, *Asian J. Math.* **11** (2007), 651–669; preprint available as `arXiv:math.DG/0611611`.

[11] H. Rosenberg: Bryant surfaces, in "The global theory of minimal surfaces in flat spaces" (Martina Franca, 1999), 67–111, Lecture Notes in Mathematics, 1775, Springer, Berlin, 2002.

[12] M. Umehara, K. Yamada: Metrics of constant curvature 1 with three conical singularities on the 2-sphere, *Illinois J. Math.* **44** (2000), 72–94.

[13] A. van der Waall: *Lamé equations with finite monodromy*, Thesis, Universiteit Utrecht, 2002.

[14] R. O. Wells, Jr.: *Differential analysis on complex manifolds*, 2nd ed., Graduate Texts in Mathematics, vol. 65, Springer-Verlag, New York, 1980.

[15] E. T. Whittaker, G. N. Watson: A course of modern analysis, Cambridge Mathematical Library, Cambridge University Press, Cambridge, 1996.